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The So-called Fibonacci Numbers in Ancient and Medieval India

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What are generally referred to as the Fibonacci numbers and the method for their formation were given by Virahāṅka (between A.D. 600 and 800), Gopāla (prior to A.D. 1135) and Hemacandra (c. A.D. 1150), all prior to L. Fibonacci (c. A.D. 1202). Nārāyaṇa Paṇḍita (A.D. 1356) established a relation between his *sṁāsikā-paṅkti*, which contains Fibonacci numbers as a particular case, and "the multinomial coefficients." © 1985 Academic Press, Inc.

Avant L. Fibonacci (env. 1202 ap. J. C.), Virahanka (entre 600 et 800 ap. J. C.), Gopala (avant 1135 ap. J. C.), et Hemacandra (env. 1150 ap. J. C.) intruidisirent les nombres de Fibonacci ainsi qu'une méthode de les générer. Narayana Pandita (1356 ap. J. C.) établit une relation entre son *samasika-pankti*, dont les nombres de Fibonacci sont un cas particulier, et les "coefficients multinomiaux." © 1985 Academic Press, Inc.

Die gewöhnlich nach Fibonacci bezeichnete Zahlenfolge wie auch deren Bildungsgesetz wurden von Virahānka (zwischen 600 und 800 nach Christus), Gopāla (vor 1135), und Hemacandra (um 1150) angegeben, die alle früher als L. Fibonacci (um 1202) lebten. Nārāyaṇa Paṇḍita (1356) fand eine Beziehung zwischen seinen *sāmāsikā-paṅkti*, worin die Fibonacci-Zahlen als Sonderfall enthalten sind, und den "multinomialen Koeffizienten." © 1985 Academic Press, Inc.

एल• फिबोनाची (सन् 1202) के पूर्व ही विरहांक (सन् 600–सन् 800 के मध्य), गोपाल (सन् 1135 के पूर्व) एवम् हेमचन्द्र (सन् 1150 के निकट) ने तथाकथित फिबोनाची संख्याओं तथा उनके निर्माण-विधि का वर्णन किया है। नारायण पंडित (सन् 1356) ने सामासिका-पंक्ति, जिसका एक खास रूप फिबोनाची संख्यायें हैं, एवम् बहुपदी गुणकों के बीच सम्बन्ध स्थापित किया है। © 1985 Academic Press, Inc.

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এম ফিবোনাচীর (1202 খৃষ্টাব্দ) পূর্বে বিরহাঙ্ক (খৃ. ৬০০-খৃ. ৪০০ মধ্যে), গোপাল (1135 খৃষ্টাব্দের পূর্বে) এবং হেমচন্দ্র (1150 খৃষ্টাব্দের নিকট) সকলেই তথাকথিত ফিবোনাচী সংখ্যা এবং তার নির্মাণ বিধির বর্ণনা করে গেছেন। নারায়ন পণ্ডিত (1356 খৃষ্টাব্দ) সামাসিক পংক্তি, যার এক বিশেষ রূপ হচ্ছে ফিবোনাচী সংখ্যা, এবং বহুপদী গুণকের মধ্যে সম্বন্ধ স্থাপন করেছেন।

INTRODUCTION

The name of Leonardo of Pisa, also called Fibonacci (1170–1250), is attached to the sequence 0, 1, 1, 2, 3, 5, 8, 13, . . . , in which the nth term is given by $U_n = U_{n-1} + U_{n-2}$ [Smith 1958, 217]. But the sequence was well known in India before Leonardo's time. Indian authorities on the metrical sciences used this sequence in works on metric.

The basic units in Sanskrit prosody are a letter having a single *mātrā* (mora or a syllabic instant) called *laghu* (light) and that having two morae called *guru* (heavy). The former is denoted by | and the latter by *S*, and their role in metric is the same as that of 1 and 2 in combinatorics.

Meters in Sanskrit and Prakrit poetry mainly fall under the following three categories: *Varṇa-vṛttas* are meters (*vṛttas*) in which the number of letters (*varṇas*) remains constant and the number of morae is arbitrary. *Mātrā-vṛttas* are meters in which the number of morae remains constant and the number of letters is arbitrary. Finally, there are meters (called *gaṇa-vṛttas*) consisting of groups (*gaṇas*) of morae such as the *aryā*, the *vaitālīya*, etc. In the latter type the number of morae in a group remains constant and the number of letters is arbitrary. However, the number of morae in different groups will, in general, be different.

A meter of the type *varṇa-vṛttas* having one letter has two variations, a *guru* and a *laghu*, and the expansion (called *prastāra*) of such a meter is obtained by writing the latter below the former, thus:

S
|.

A meter of this kind having two letters has four variations. The expansion of such a meter is obtained by writing, in order, each variation in the former expansion on the left of a *guru*, and then the same on the left of a *laghu*:

S S
| S
S |
| |.

The same process of expansion is followed for meters having more than two letters. Thus, the expansion of a meter having n letters is obtained by writing, in order, each variation in the expansion of a meter having $(n - 1)$ letters on the left of a *guru*, and then the same on the left of a *laghu*.

A similar expansion process is applicable to *mātrā-vṛttas*. Here, the number of variations of a meter of one mora is 1 and its expansion is written as |. The number of variations of a meter of 2 morae is 2, and its expansion is written as

S
| |.

The number of variations of a meter of 3 morae is 3 and its expansion is obtained by writing the variation of the meter of one mora on the left of a *guru* and then writing, in order, each variation in that of 2 morae on the left of a *laghu* (see Table I). The same process of expansion is followed for meters having more than 3 morae. Thus, the expansion of a meter of n morae is obtained by writing, in order, each variation in the expansion of a meter of $(n - 2)$ morae, on the left of a *guru*, and then, the same of $(n - 1)$ morae on the left of a *laghu*. Table I, in which the

TABLE I

<table>
<tr><th>1 mora</th><th>2 morae</th><th>3 morae</th><th>4 morae</th><th>5 morae</th></tr>
<tr><td>|</td><td>S
| |</td><td>| S
S |
		</td><td>S S 		S 	S	 S	
			</td><td>	S S S	S 			S S S
	S
S		 S		
			</td></tr>
</table>

TABLE II

(1) S S S	(6) | S S |	(10) | | S | |
(2) | | S S	(7) S | S |	(11) | S | | |
(3) | S | S	(8) | | | S |	(12) S | | | |
(4) S | | S	(9) S S | |	(13) | | | | | |
(5) | | | | S		

number of morae in a meter is written above and its expansion below, illustrates this rule.

The expansion of the meter having 6 morae, along with the serial number of each variation, is as given in Table II.

The expansion of *mātrā-vṛttas* just described corresponds to a partitioning of a number (the number being the number of morae in the meter), where the digits take on the values 2 and 1 and their order is relevant; the number of digits in a partition, however, is arbitrary.

Here, it is easily seen that the variations of *mātrā-vṛttas* form the sequence of numbers which are now called Fibonacci numbers. For, the numbers of variations of meters having 1, 2, 3, 4, 5, 6, . . . morae are, respectively, 1, 2, 3, 5, 8, 13, . . . , and these are the Fibonacci numbers. It is also observed that the method for finding the numbers of variations of *mātrā-vṛttas* leads to the general rule, $U_n = U_{n-1} + U_{n-2}$ for the formation of Fibonacci numbers. Thus it can be safely concluded that the concept of the sequence of these numbers in India is at least as old as the origin of the metrical sciences of Sanskrit and Prakrit poetry.

EARLY DEVELOPMENTS: ĀCĀRYA PIṄGALA AND ĀCĀRVA BHARATA

Ācārya Piṅgala is the first authority on the metrical sciences in India whose writings indicate a knowledge of the so-called Fibonacci numbers. In his commentary, the *Vedārthadīpikā* on *Ṛksarvānukramaṇī*, Sadguruśiṣya writes that Ācārya Piṅgala was a younger brother of Ācārya Pāṇini [Agrawala 1969, 16]. There is an alternative opinion that he was a maternal uncle of Pāṇini [Vinayasagar 1965, Preface, 12]. The period during which Pāṇini was active has been discussed by several scholars (such as A. A. Macdonell, A. Weber, G. A. Grierson, T. Goldstucker, and others) who have, for different reasons, placed him between 700 B.C. and A.D. 100. Agrawala [1969, 463–476], after a careful investigation in which he considered the views of earlier scholars, has concluded that Pāṇini lived between 480 and 410 B.C.

According to Yādava [Sinharay 1977, 105], a commentator belonging to the 10th century, Piṅgala's rule "*miśrau ca*" (i.e., "and the two mixed")[1] is also meant for the expansion of *mātrā-vṛttas*, and *mātrā-vṛttas* should be expanded by combining the expansions of two earlier meters with a *guru* and a *laghu*, respectively.

[1] Throughout the paper the translations of the texts are by the author of the paper.

According to Yādava [Sinharay 1977, 41] and another commentator, Halāyudha [Madhusudana 1981, 46], who also belonged to the 10th century, Piṅgala's rule "Gau gantamadhyādirlaśca" gives the variations of a *gaṇa* (i.e., a group of letters, the group having 4 morae) in an *āryā* meter. Following these commentators the rule may be translated as "Two *gurus*, a *guru* (in) the end, the middle, the beginning and (all) *laghus*." Here, the variations have been stated in the order in which they are obtained by the expansion rule given above.

Ācārya Bharata is the next authority on the metrical sciences whose writings indicate a knowledge of the so-called Fibonacci numbers. Various dates between 100 B.C. and A.D. 350 have been assigned by different scholars to *Nātyaśāstra* of Ācārya Bharata. Though the work has not come down to us intact, Kane [1961, 40–47], after discussing the various possibilities, assigns the work a date earlier than A.D. 300 and not much older than the beginning of the Christian era.

According to Abhinavagupta, whose literary activities ranged between A.D. 980 and 1020 [Kane 1961, 243], Ācārya Bharata's rule [Shastri 1975, 1192] for the expansion of *mātrā-vṛttas* is ". . . *miśrau cetyapi mātrikau*." This may be translated as ". . . 'And the two mixed' is also (meant) for *mātrā(-vṛttas)*." Again, it is to the process of mixing the expansions of two earlier meters with *gurus* and *laghus*, respectively, that this quotation refers.

In another Hindu work, *Viṣṇudharmottara Purāṇa*, a chapter on Sanskrit prosody indicates a knowledge of the so-called Fibonacci numbers.

According to Bühler [Gairola 1960, 231–232] the *Viṣṇudharmottara Purṇa* was probably written in Kashmir during the seventh century A.D. Verse 16 (Pt. 3, Chap. iii) of the *Purāṇa* [Shah 1958] contains the following rule for the expansion of *mātrā-vṛttas*: ". . . *mātrācchandānstathaiva hi. uktavarṇākṣaram* (?*ra*) *cchando bhavedekavivarjitah* . . . ," which may be translated as ". . . Leaving one in the beginning (the process of expansion) of *mātrā-vṛttas* is similar to (that of) *varṇa-vṛttas*, as stated above." We have already seen that this process of expansion of *mātrā-vṛttas* is similar to that of *varṇa-vṛttas*. However, the expansion of a *mātrā-vṛtta* (having one mora) differs from that of a *varṇa-vṛtta* (having one letter).

SO-CALLED FIBONACCI NUMBERS IN INDIA PRIOR TO A.D. 1200: ĀCĀRYA VIRAHĀṄKA, GOPĀLA AND ĀCĀRYA HEMACANDRA

Ācārya Virahāṅka is the first authority on the metrical sciences who explicitly gave the rule for the formation of numbers of variations of *mātrā-vṛttas* (the so-called Fibonacci numbers).

Velankar [1962a, Introduction, xxi–xxv] has given an extensive analysis of the activities of Ācārya Virahāṅka, concluding finally that Ācārya Virahāṅka lived some time between the sixth and eighth centuries. An English translation of Velankar's translation (Sanskrit) of the rule (Prakrit) [1962a, 101] follows: "The variations of two earlier meters being mixed, the number is obtained. That is a direction for knowing the number (of variations) of the next *mātrā(-vṛtta)*."

Gopāla is the first author on metric who specifically mentions the numbers of variations of *mātrā-vṛttas* (the so-called Fibonacci numbers) while commenting on this rule of Ācārya Virahāṅka. According to Velankar [1962a, Introduction, xxvii] a palm leaf manuscript of Gopāla's commentary on *Vṛttajātisamuccaya* of Ācārya Virahāṅka, handwritten between A.D. 1133 and 1135, is available at Jesalmere. The manuscript forms the basis of this edition of the *Vṛttajātisamuccaya*. Gopāla discusses Virahāṅka's rule in some detail:

> Variations of two earlier meters [is the variation] of a *mātrā-vṛtta*.
>
> For example, for [a meter of] three [morae], variations of two earlier meters, one and two, being mixed three happens.
>
> For [a meter] of four [morae], variations of meters of two morae [and] of three morae being mixed, five happens.
>
> For [a meter] of five [morae] variations of two earlier [meters] of three morae [and] of four morae, being mixed, eight is obtained.
>
> In this way, for [a meter] of six morae, [variations] of four morae [and] of five morae being mixed, thirteen happens. And like that, variations of two earlier meters being mixed, [variations of a meter] of seven morae [is] twenty-one.
>
> In this way, the process should be followed in all *mātrā-vṛttas*. [Velankar 1962a, 101]

Ācārya Hemacandra, one of the greatest Jain writers, is the other authority on metric who specifically mentions the numbers of variations of *mātrā-vṛttas* in his *Chandonuśāsana* (c. 1150). He lived at Anhilvad Patan in Gujrat and enjoyed the patronage of two kings, Siddharāja and Kumārapāla. Bühler states that Ācārya Hemacandra wrote *Chandonuśāsana* and its commentary between A.D. 1142 and 1158 [Banthia, 1967, Preface, 16–27, 57–60]. Ācārya Hemacandra's rule [Velankar 1961, 239] may be translated as follows: "Sum of the last and the last but one numbers [of variations] is [that] of the *mātrā-vṛtta* coming next." Ācārya Hemacandra also comments on his own rule, as did Gopāla on Virahāṅka's rule, concluding with "Statement—1, 2, 3, 5, 8, 13, 21, 34 and in this way, afterwards" [Velankar 1961, 239].

The rule for the formation of numbers of variations of *mātrā-vṛttas* continued to be given in works on metric even after A.D. 1200. *Kavidarpaṇa* is such a work on metric which deals with the topic. Velankar advances reasons to establish that *Kavidarpaṇa* was composed during the 13th century A.D. [1962b, Introduction, iv]. The author of *Kaidarpaṇa* states a rule [Velankar 1962b, 67–68] for the formation of numbers of variations of *mātrā-vṛttas* that is essentially the same as that given by Hemacandra.

Thus we see that the sequence of the so-called Fibonacci numbers resulted as a natural consequence of the variations of *mātrā-vṛttas* of 1, 2, 3, . . . , morae.

THE PRĀKṚTA PAIṄGALA

Prākṛta Paiṅgala is another important work on metric that gives several rules regarding the so-called Fibonacci numbers. The period of composition of *Prākṛta Paiṅgala* has been discussed by Vyasa, who concludes that *Prākṛta Paiṅgala* was written during the first quarter of the 14th century A.D. [1962, 6–20]. Probably the present form of *Prākṛta Paiṅgala* is an amplification of an old work [Velankar 1962a, Introduction, xiv].

In connection with the so-called Fibonacci numbers, the author of *Prākṛta Paiṅgala* gives rules regarding *naṣṭa* and *uddiṣṭa* analyses of *mātrā-vṛttas*. These are the usual combinatorial problems discussed by the Hindus in metric and are related to measurement of morae and letters in verses. The aim of the *naṣṭa* analysis is to find *naṣṭa-bheda* (the unknown structure) of a particular variation of a given meter, the *uddiṣṭāṅka* (the serial number of this variation from among all variations of the meter) being given. Conversely, the aim of the *uddiṣṭa* analysis is to find *uddiṣṭāṅka* of a variation amidst all variations of the meter, the *naṣṭa-bheda* of the variation being given.

The author of *Prākṛta Paiṅgala* gives the following rule for finding *naṣṭa-bheda* corresponding to a given *uddiṣṭāṅka*:

> In *naṣṭa(-bheda)* make all morae, *laghus*. Give [i.e., write] the number equal to the pair of numbers, overhead. Omit [i.e., subtract] the number asked [i.e., given]. Write the remainders after subtractions and omit the remaining numbers. Piṅgala Nāga says that *laghus* of the parts takes *laghus* of others [i.e., of those lying ahead] to become *gurus* [and] writing thus, the writing [i.e., the structure] comes immediately. [Vyasa 1959, 32]

Let us consider the fourth variation of the meter having 6 morae. Making all morae *laghus* and writing the number equal to the pair of numbers (i.e., 1, 2, 3, 5, 8, 13, . . .) above, we obtain

```
1 2 3 5 8 13
| | | | |  | .
```

The number 4 (of the variation) is subtracted from 13, giving 9. Next the numbers written above are subtracted from 9, and then from the remainders, repeatedly. Thus we have $9 - 8 = 1$ and $1 - 1 = 0$. *Laghus* below the subtracted numbers (i.e., below 8 and 1) combine with the next *laghus* to give *gurus*. Other *laghus* remain as they are. This gives the structure of the desired fourth variation to be *S* | | *S* (see Table II).

The author of *Prākṛta Paiṅgala* gives the following rule for finding *uddiṣṭāṅka* corresponding to a given *naṣṭa-bheda*: "Write the number equal to the pair of numbers, above. Omit all the numbers (except those) over *gurus*. Having written, bring the remainder. Know that as *uddiṣṭa*" [Vyasa 1959, 31].

The *uddiṣṭāṅka* may be found for the structure as given above. Writing the numbers equal to the pair of numbers (i.e., 1, 2, 3, 5, 8, 13, . . .) above (and also below *gurus*, in order) we obtain

```
1 3 5  8
S | |  S
2     13.
```

Subtracting the numbers over *gurus* (i.e., 1 and 8) from the number of variations of the meter (i.e., from 13), 4 is obtained as the remainder; this is the desired *uddiṣṭāṅka* of the variation whose structure is given (see Table II).

The next rule related to the so-called Fibonacci numbers, as given by the author of *Prākṛta Paiṅgala*, is meant for finding the serial numbers of variations of a meter having a particular number of *gurus* (or *laghus*) among serial arrangement of all variations of the meter. The rule may be translated as follows: "Keep the numbers like [those in] *uddiṣṭa*. Subtract the numbers, leftwards. Bring a single guru [for] single subtraction. Know double, triple *gurus* [for] double, triple subtractions" [Vyasa 1959, 43–44].

To illustrate the rule, consider the variations of a meter having 6 morae, letting the variations be arranged serially (see Table II). Obviously, 13 gives the serial number of the variation having no *guru* (see Table II). The numbers of variations of meters having 1, 2, 3, 4, and 5 morae (i.e., 1, 2, 3, 5, and 8) are subtracted from the number of variations of the meter having 6 morae (i.e., from 13), in the reverse order. This gives 5, 8, 10, 11, and 12, which are the serial numbers of the variations having a single *guru*, amidst all variations of the meter (see Table II). Again the same numbers (i.e., 1, 2, 3, 5, 8, . . .) are subtracted from each of the remainders, separately. Leaving the numbers that appeared earlier as remainders gives 2, 3, 4, 6, 7, and 9, which are the serial numbers of the variations having two *gurus* (see Table II). The process is repeated, giving 1, the serial number of the variation having three *gurus* (see Table II).

We have already seen the correspondence between partitions of a number (the number being the number of morae in the meter)—where the digits take on the values 2 and 1, their order being relevant and the number of digits in a partition being arbitrary—and the expansion of *mātrā-vṛttas*. Correspondingly, the rule gives the serial numbers of p-partitions of n (i.e., partitions containing p digits each, where n is the number of morae in the meter for $p = n, n - 1, \ldots$ amidst all partitions of n).

Still more interesting is the rule given in Prakrit by the author of *Prākṛta Paiṅgala* for the formation of *matta-meru*, or *mātrā-meru*, (*meru*-like table for *mātrāvṛttas* where *meru* is the name of the imaginary mountain that is supposed to stand at the center of the earth). It gives a method for determining the numbers of variations (of a meter) having a definite number of *gurus* (or *laghus*) from among the variations of the meter. It also establishes a relation between the sequence of binomial coefficients and the (so-called) Fibonacci numbers, thus providing an alternative method for the formation of these numbers. On the basis of the Hindi commentary, the text [Vyasa 1959, 41–42] may be analyzed. Initially (two rows of) two cells (each) are formed. (Next, two rows of 3, 4, etc., cells each are formed.) Unity is (written) in the last cells of these (rows). Again unity is (written) in the first cells (of the first, the third, the fifth) . . . , rows and 2, 3, 4, . . . , in the first cells of the second, the fourth, the sixth, . . . , rows, in order. Every other cell is filled by (the sum of) the number lying above (that cell or above that cell to the left, added to) the number above the (latter number and to its right). *Mātrā-meru* for a meter of 7 morae and formed according to this rule is shown in Table III. Here unities have been kept in the last cells of all the rows. For an integer m, unities have been kept in the first cells of the $(2m - 1)$st rows, and $m + 1$ in those of the $(2m)$th rows, $m = 1, 2, 3, \ldots$. Thus, 1 has been kept in each of the first cells of the first, third, and fifth rows, while 2, 3, and 4 are in those of the

TABLE III

dvikala	1	1			= 2	(2 morae)
trikala	2	1			= 3	(3 morae)
catuṣkala	1	3	1		= 5	(4 morae)
pañcakala	3	4	1		= 8	(5 morae)
ṣaṭkala	1	6	5	1	= 13	(6 morae)
saptakala	4	10	6	1	= 21	(7 morae)

second, fourth, and sixth rows. In the remaining cases, a cell of the $(2m - 1)$st row contains the sum of the number lying above it and to the left with the number in the cell above the latter cell and to the right. Thus, for the second cell in the third row and the second and third cells in the fifth row, we have $2 + 1 = 3, 3 + 3 = 6$, and $4 + 1 = 5$ as their entries; for a cell in the $(2m)$th row we use the sum of numbers in the cell just above it and in the cell above the latter cell and to the right. Thus the numbers in the second cell of the fourth row and the second and third cells of the sixth row are $3 + 1 = 4, 6 + 4 = 10$, and $5 + 1 = 6$.

It will be observed that the numbers in the cells of the $(n - 1)$th row, from right to left, are the numbers of those variations of the meter of n morae having 0, 1, 2, . . . *gurus*, respectively, and the sum of the numbers in that row is the number of variations of the meter. This implies that these are the numbers of p-partitions of n, for $p = n, n - 1, \ldots$, and that their sum is the number of all partitions of n.

The *mātrā-meru* also establishes a relation between the sequence of binomial coefficients and the (so-called) Fibonacci numbers. Let U_n denote the number of variations of a meter having n morae. Expressing n as $2m$ or $2m + 1$, according to whether it is even or odd, then, as is clear from the meru,

$$U_n = \sum_{r=0}^{m} {}^{n-r}C_r .$$

This gives another method for the formation of the so-called Fibonacci numbers. Here, it is also seen that numbers in a cell of the *mātrā-meru* are the binomial coefficients (or their sum) formed according to a definite rule.

FURTHER DEVELOPMENTS: THE GAṆITA KAUMUDĪ OF NĀRĀYAṆA PAṆḌITA

The *Gaṇita Kaumudī* was the first mathematical work in which the ideas of the so-called Fibonacci numbers were developed further. From the colophon toward the end of the book we learn that the *Gaṇita Kaumudī* was written by Nārāyaṇa Paṇḍita in A.D. 1356 [Dvivedi 1942, 411]. In Chapter 13 of the book the author defines *sāmāsikā-paṅkti* (additive sequence). The (so-called) Fibonacci numbers are a particular case of this *paṅkti* (sequence).

Nārāyaṇa's *sāmāsikā-paṅkti* is his tool for the treatment of permutations, combinations, partitions, etc.; it is defined with respect to "the greatest digit." For a particular permutation, combination, or partition, we have the following correspondences between Nāryāṇa's descriptions and contemporary language and notation: the digits from one to "the greatest digit" ($= q$); the "number of places" is simply the number of digits ($= p$); the "sum of digits" ($= n$) is just that.

Nārāyaṇa's rule for the formation of *sāmāsikā-paṅkti* may be translated as follows:

> First keeping unity twice, write their sum ahead. Write ahead of that, the sum of numbers from the reverse order [and in] places equal to the greatest digit. In the absence of [numbers in] places equal to the greatest digit, write ahead the sum of those [in available places]. Numbers at places [equal to] one more than "the sum of digits" happen to be "the *sāmāsikā-paṅkti*." [Dvivedi 1942, 290–291]

Let $v(q,r)$ denote the rth term of the *sāmāsikā-paṅkti* when "the greatest digit" is q. According to the rule, $v(q,1) = 1$ and $v(q,2) = 1$. For other values of r, if $3 \leq r \leq q$, $v(q,r) = v(q,r-1) + v(q,r-2) + \cdots + v(q,2) + v(q,1)$; and if $q < r$, $v(q,r) = v(q, r-1) + v(q, r-2) + \cdots + v(q, r-q)$. The sequence is finite having $(n+1)$ terms, n being "the sum of digits."

For $q = 2$ we obtain the sequence 1, 1, 2, 3, 5, 8, . . . , which are the so-called Fibonacci numbers. Therefore, these numbers are a particular case of the *sāmāśikā-paṅkti* of Nārāyaṇa Paṇḍita.

The relation between the so-called Fibonacci numbers and the binomial coefficients as contained in the *Prākṛta Paiṅgala* has already been established with the aid of the figure of numbers, the *mātrā-meru*. It has also been observed that the numbers in the cells of a *mātrā-meru* are either the binomial coefficients or their sum, formed according to a definite rule. Similarly, Nārāyaṇa Paṇḍita established a relation between his *sāmāsikā-paṅkti* and the multinomial coefficients. Here, the relation is established with the help of another figure of numbers called *matsya-meru* (fish-*meru*). Like *mātrā-meru,* numbers in cells of a *matsya-meru* form "the sequence of multinomial coefficients," or *sūcī-paṅkti* (needle-like sequence).

Sūcī-paṅkti is defined with respect to "the number of places" ($= p$) and "the greatest digit" ($= q$). It is formed with the help of *vaiśleṣiṇī-paṅkti* (sequence of separated units) of measure q, which consists of a sequence of q 1's. Nārāyaṇa's rule for the formation of *sūcī-paṅkti* follows.

> Put the *vaiśleṣiṇī-paṅkti* of measure [equal to] the greatest digit, separately, in places equal to the number of places. Their [final] product is the *sūcī-paṅkti* [needle-like sequence]. [Dvivedi 1942, 295]

Thus for $p = 3$ and $q = 3$ the *vaiśleṣiṇī-paṅkti* of measure 3 is 1, 1, 1. Keeping *vaiśleṣiṇī-paṅkti* at 3 places and multiplying successively, we get 1, 3, 6, 7, 6, 3, and 1 to be the final products (see Table IV). These products make up the desired *sūcī-paṅkti* when $p = 3$ and $q = 3$.

The *vaiśleṣiṇī-paṅkti* of measure q (i.e., q 1's) are the coefficients of x^r in the polynomial $1 + x + x^2 + \cdots + x^{q-1}$, $r = 0, 1, \ldots, q-1$. Therefore, from the

TABLE IV
FORMATION OF *SŪCĪ-PAṄKTI*
FOR $p = 3$ AND $q = 3$

1,1,1	(1st place)
1,1,1	(2nd place)
1,1,1	
1,1,1	
1,1,1	
1,2,3,2,1	
1,1,1	(3rd place)
1,2,3,2,1	
1,2,3,2,1	
1,2,3,2,1	
1,3,6,7,6,3,1	

method of formation of the *sūcī-paṅkti* it is clear that when the greatest digit is q and the number of places is p, then the $(r + 1)$st term of the sequence, say $u(p, q, r)$, is the multinomial coefficient of x^r in the expansion of $(1 + x + x^2 + \cdots + x^{q-1})^p$, $r = 0, 1, 2, \ldots, (q - 1)p$.

From the process of addition of the products in this method of obtaining the *sūcī-paṅkti*, an alternative method of formation is also apparent. Let two such sequences be formed with respect to the same greatest digit q, but for a different number of places. Assuming that the number of places for the first sequence is $p - 1$ and p for the second, any term of the second sequence can be obtained from those of the first. Consider the $(r + 1)$st term of the second sequence. If $r + 1 \leq q$, then this term of the second sequence is equal to the sum of the first r terms of the first sequence. If, however, $q < r + 1 \leq (p - 1)(q - 1) + 1$, this term is equal to the sum of q terms, up to the $(r + 1)$st term of the second sequence. Finally, if $(r + 1) > (p - 1)(q - 1) + 1$, then the $(r + 1)$st term of the second sequence is the sum of the terms of the first sequence that follow the $(r - q + 1)$st term.

Nārāyaṇa makes use of this method of formation of the *sūcī-paṅkti* (needle-like sequence) in the formation of his figure of numbers named *matsya-meru*. The figure is formed with respect to the greatest digit q and the sum of digits n. His rule for the formation of *matsya-meru* follows.

> [Form] one [cell] initially [and then] lines of cells, [the cells in a line] increasing by "one less than the greatest digit," till the measure of that is one more [than the sum of digits]. Leaving the first cell and starting below the second, form horizontal lines [of cells] equal to the sum of digits.
>
> After that, putting [one] initially, keep unity in the [initial] stretched line. [Take cells] above one's own cell in the reverse order and equal to the greatest digit. Write the sum of numbers [in them] in the lower cell. If there be an absence [of cells] equal to the greatest digit, [take] the sum of numbers as far as possible. Work, in order. [Dvivedi 1942, 304–305]

To illustrate the rule, Nārāyaṇa considers the case when $q = 3$ and $n = 7$,

TABLE V

MATSYA-MERU FOR $q = 3$ AND $n = 7$[a]

1										$p = 0$
	1	1	1							$p = 1$
		1	2	3	2	1				$p = 2$
			1	3	6	7	6	3	1	$p = 3$
				1	4	10	16			$p = 4$
					1	5	15			$p = 5$
						1	6			$p = 6$
							1			$p = 7$

[a] The figure of *matsya-meru* given in the printed text seems to be vitiated, for it does not satisfy the rules [Dvivedi 1942, 305]. The figure shown here is based upon a manuscript of *Gaṇita Kaumudī*, available at Sampurnanand Sanskrit University, Varanasi [Manuscript No. 35668/p. 7 recto].

obtaining the numbers of cells in a line as 1, 3, 5, and 7. By the above process he arrives at the *matsya-meru* as shown in Table V.

As for the relation between terms of a *sāmāsikā-paṅkti* and those of a *sūcī-paṅkti* (i.e., multinomial coefficients), we see that for all values of p and q, the first term of a *sāmāsikā-paṅkti* is equal to the first term of a *sūcī-paṅkti*, each being equal to 1. For other values of p and q, we observe that $v(q, 2) = u(1, q, 0)$ and $v(q, 3) = u(2, q, 0) + u(1, q, 1)$, both sides of these equalities being equal to 1 and 2, respectively. Similarly, $v(q, 4) = u(3, q, 0) + u(2, q, 1) + \cdots$, and so on. In general, $v(q, t + 1) = u(t, q, 0) + u(t - 1, q, 1) + \cdots + u(t - k, q, k)$, where $k = p + r - t$ and $t \leq (p + r)/q < t + 1$. A comparison of these relations with the *matsya-meru* (when the greatest digit is q) shows that the right sides corresponding to the rth relation yield the numbers in cells of the rth vertical line of the *matsya-meru*. Nārāyaṇa's rule for this case is precisely this: "*Sāmāsikā-paṅkti* is formed by the sum of numbers in vertical cells of that" [Dvivedi 1942, 305]. Thus we have another method for the formation of *sāmāsikā-paṅkti*, of which the so-called Fibonacci numbers are a particular case.

It has already been shown that the numbers of variations of *mātrā-vṛttas* are the so-called Fibonacci numbers. It has also been observed that the variations of *mātrā-vṛttas* correspond to a partitioning of a number where the digits take on the values 2 and 1 and their order is relevant. So, there exists a relation between the so-called Fibonacci numbers and such partitions. Since the former constitute a particular case of *sāmāsikā-paṅkti*, there must exist partitions of a number corresponding to the *sāmāsikā-paṅkti*. Nārāyaṇa discusses such partitions of a number and establishes several relations between the two. For $q \leq 10$, his partitions contain all the digits between 1 and q. Nārāyana's rule for expansion of such partitions is described next [Dvivedi 1942, 339]. In the first partition the greatest digit is written in all places and, if required, on their left that digit is written which when added yields the given number. Starting from the beginning each time, the first available digit different from unity is replaced by the next lesser digit. Other digits on the right of the replaced digit are written as described above. Clearly, the digits on the left of a replaced digit are all q, but they may be any lesser digit, if at the beginning.

Nārāyaṇa illustrated his rule with the help of two expansions in which $n = 7$ and $p \leq 7$ [Dvivedi 1942, 340]. In the first expansion (see Table VI), $q = 7$, whereas in the second expansion (Table VII), $q = 3$. In all cases we observe that the number of partitions of $n = v(q, n + 1)$ and the number of partitions ending in 1, 2, . . . are equal to $v(q, n + 1 - t)$, where $t = 1, 2, \ldots$. Here Nārāyaṇa's rule is, "Last number of *sāmāsikā-paṅkti* is the number of variations. Starting from the penultimate, numbers in the reverse order are the numbers of variations ending in 1, etc." [Dvivedi 1942, 335]. Thus in the first example, where $q = 7$, the *sāmāsikā-paṅkti* is 1, 1, 2, 4, 8, 16, 32, 64; since $n = 7$, the number of partitions of 7 is 64, and the numbers of partitions of 7 ending in 1, 2, . . . , 7 are, starting from the penultimate in the reverse order, 32, 16, . . . , 1. Similarly, since $q = 3$ in the second example, the *sāmāsikā-paṅkti* is 1, 1, 2, 4, 7, 13, 24, 44, and $n = 7$ implies that the number of partitions of 7 is 44 and the numbers of partitions of 7 ending in 1, 2, 3 are, starting from the penultimate in the reverse order, 24, 13, and 7. Here

TABLE VI
SERIAL ARRANGEMENT OF VARIATIONS FOR $n = 7$, $p \leq 7$, AND $q = 7$[a]

(1)	7	(17)	52	(33)	61	(49)	511
(2)	16	(18)	142	(34)	151	(50)	1411
(3)	25	(19)	232	(35)	241	(51)	2311
(4)	115	(20)	1132	(36)	1141	(52)	11311
(5)	34	(21)	322	(37)	331	(53)	3211
(6)	124	(22)	1222	(38)	1231	(54)	12211
(7)	214	(23)	2122	(39)	2131	(55)	21211
(8)	1114	(24)	11122	(40)	11131	(56)	111211
(9)	43	(25)	412	(41)	421	(57)	4111
(10)	133	(26)	1312	(42)	1321	(58)	13111
(11)	223	(27)	2212	(43)	2221	(59)	22111
(12)	1123	(28)	11212	(44)	11221	(60)	112111
(13)	313	(29)	3112	(45)	3121	(61)	31111
(14)	1213	(30)	12112	(46)	12121	(62)	121111
(15)	2113	(31)	21112	(47)	21121	(63)	211111
(16)	11113	(32)	111112	(48)	111121	(64)	1111111

[a] Omissions in variations numbering 4, 15, 49, and 54, found in [Dvivedi 1942, 340], have been completed with the help of the manuscript of *Gaṇita Kaumudī* [Manuscript No. 35668/p. 16 recto], referred to in Table V. Similarly, variation No. 27 has also been corrected. Variation No. 30 has been changed so that it satisfies the rules.

TABLE VII
SERIAL ARRANGEMENT OF VARIATIONS FOR $n = 7$, $p \leq 7$, AND $q = 3$[a]

(1)	133	(12)	2122	(23)	2131	(34)	3211
(2)	223	(13)	11122	(24)	11131	(35)	12211
(3)	1123	(14)	1312	(25)	1321	(36)	21211
(4)	313	(15)	2212	(26)	2221	(37)	111211
(5)	1213	(16)	11212	(27)	11221	(38)	13111
(6)	2113	(17)	3112	(28)	3121	(39)	22111
(7)	11113	(18)	12112	(29)	12121	(40)	112111
(8)	232	(19)	21112	(30)	21121	(41)	31111
(9)	1132	(20)	111112	(31)	111121	(42)	121111
(10)	322	(21)	331	(32)	2311	(43)	211111
(11)	1222	(22)	1231	(33)	11311	(44)	1111111

[a] Variation No. 15 in [Dvivedi 1942, 340] has been corrected in this table on the basis of the manuscript of *Gaṇita Kaumudī* [Manuscript No. 35668/p. 16 recto] referred to earlier. Variation No. 30 has been changed so that it satisfies the rules.

we also note that if $q = 2$, the partitioning is the same as that corresponding to the variations of *mātrā-vṛttas*, and the *sāmāsikā-paṅkti* (as observed earlier), become the so-called Fibonacci numbers.

The *naṣṭa-bheda* and *uddiṣṭāṅka* for such partitions are obtained with the help of *unmeru* (elevated-*meru*), another kind of figure of numbers. Nārāyaṇa's rule for the formation of *unmeru* is described next [Dvivedi 1942, 341–342]: One cell is formed in the first row. The number of cells increases by one in each subsequent row until the number of cells in a row is one more than the sum of the digits. The *sāmāsikā-paṅkti* is written in the cells of the lowest row; natural numbers, in the reverse order, appear in the cells of the remaining rows. Naturally, the numbers in the columns are also natural numbers. The numbers greater than "the greatest digit" are omitted.

To illustrate his rule Nārāyaṇa considered the case when $n = 7$ and $q = 3$, obtaining the figure shown in Table VIII.

Nārāyaṇa's rule for finding the *naṣṭa-bheda* corresponding to a given *uddiṣṭāṅka* follows.

> Subtract the *naṣṭāṅka* from the last number of the *sāmāsikā-paṅkti*. Subtract from the remainder, the numbers nearest to the remainders until no subtraction is possible. The digit obtained in the cell common to the horizontal line and the vertical line is the digit of the *naṣṭa*[-*bheda*]. [Write] those numbers in a series. In this way, in [finding] *naṣṭa*[-bheda], work this process dependent upon *unmeru*, repeatedly. [Dvivedi, 1942, 343]

The method seems to be similar to that for finding the *naṣṭa-bheda* corresponding to a given *uddiṣṭāṅka* as contained in the *Prākṛta Paiṅgala* and meant for *mātrā-vṛttas*. In such a case the horizontal line (row) and the vertical line that determine a common cell are chosen as follows: In the beginning the row is just above the row containing the *sāmāsikā-paṅkti*. Later on it is just above the row containing the uppermost cell of the column whose digit has been already noted. The column is the column whose digit could not be subtracted. Moreover, the series is to be formed from right to left.

To clarify this rule, we will find the *naṣṭa-bheda* when the *uddiṣṭāṅka* is 36, $n = 7$, and $q = 3$. Here, the *sāmāsikā-paṅkti* will be 1, 1, 2, 4, 7, 13, 24, 44. Subtracting 36 from 44, the last number of the *sāmāsikā-paṅkti*, 8 is obtained. The next number of the *sāmāsikā-paṅkti* to the left of 44 is 24, which cannot be subtracted from the remainder 8. The digit of the *unmeru* in the row above the one containing *sāmāsikā-paṅkti* and in the same column as 24 is 1, which is written separately. Again, the next number of the *sāmāsikā-paṅkti* to the left of 24 is 13, which also cannot be subtracted from 8. The digit of the *unmeru* common to the column containing 13 and the row which is just above the uppermost cell in the column of 24 (for which 1 has already been computed as the digit of the *unmeru*) is 1; this value is situated to the left of the 1 obtained earlier. Next, the number of the *sāmāsikā-paṅkti* to the left of 13 is 7, and 7 subtracted from the remainder 8 yields 1. The next number of the *sāmāsikā-paṅkti*, to the left of 7, is 4, which cannot be subtracted from the remainder 1. The digit of the *unmeru* common to the column of 4 and the row which is just above the column of 13 is 2, which is placed to the left of 11 (as noted above). Again, the number in the *sāmāsikā-paṅkti* to the left of 4 is 2, which cannot be subtracted from the remainder 1. The digit of the *unmeru* common to the column of 2 and the row just above the column of 4 is 1, which is placed

TABLE VIII
THE *UNMERU* FOR $n = 7$ AND $q = 3$[a]

1							
2	1						
3	2	1					
	3	2	1				
		3	2	1			
			3	2	1		
				3	2	1	
1	1	2	4	7	13	24	44

[a] The figure in [Dvivedi 1942, 34] also contains some errors and has been corrected on the basis of the manuscript of *Gaṇita Kaumudī* [Manuscript No. 35668/p. 16 verso], where the first vertical line of the elevated part is absent. The second vertical line, though absent in the book, appears in the manuscript. Both the book and the manuscript give the *samasika-pankti* as 1, 2, 3, . . . for 1, 1, 2, . . . (in the table).

to the left of 211 (determined in the preceding step). The number in the *sāmāsikā-paṅkti* to the left of 2 is 1 and 1 − 1 = 0. Finally, the number in the *sāmāsikā-paṅkti* to the left of 1 is 1, which cannot be subtracted from 0. The digit of the *unmeru* common to the column of this 1 and the row just above the column of 2 is 2, which is written to the left of 1211; thus the desired partition is determined to be 21211.

The *uddiṣṭāṅka* corresponding to a given *naṣṭa-bheda* is obtained by a process that is the reverse of that meant for *naṣṭa-bheda* [Dvivedi 1942, 344].

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